MASSACHUSETTS INSTITUTE OF TECHNOLOGY

RE-47

IMPROVEMENT OF THE SPHEROIDAL METHOD FOR ARTIFICIAL SATELLITES

by

John P. Vinti

July 1968

EXPERIMENTAL ASTRONOMY LABORATORY
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
CAMBRIDGE 39, MASSACHUSETTS

RE-47

IMPROVEMENT OF THE SPHEROIDAL METHOD FOR ARTIFICIAL SATELLITES

by

John P. Vinti

July 1968

Experimental Astronomy Laboratory

Massachusetts Institute of Technology
Cambridge, Massachusetts

Approved Winston R. Markey
Director (RGS)
Experimental Astronomy Laboratory

ACKNOWLEDGEMENT

This report was prepared under the sponsorship of the Electronics Research Center of the National Aeronautics and Space Administration through NASA Grant NGR-22-009-262.

The publication of this report does not constitute approval by the National Aeronautics and Space Administration of the findings or the conclusions contained therein. It is published only for the exchange and stimulation of ideas.

Improvement of the Spheroidal Method for
Artificial Satellites

John P. Vinti

Experimental Astronomy Laboratory
Massachusetts Institute of Technology
Cambridge, Massachusetts

ABSTRACT

Objections to applying the spheroidal method to calculate a polar orbit of an artificial satellite are easily overcome.

Previous papers have already treated the behavior in an exactly polar orbit of the right ascension ϕ, the coordinate for which the difficulty supposedly occurs. Just as in the Keplerian problem, it remains constant, except for jumps of 180° at a pole.

There remains the case of an almost polar orbit, for which the calculation of ϕ may be inaccurate near a pole, unless one takes special precautions. The present paper first simplifies the expression for ϕ for all orbits, polar or not, and then shows how to avoid the difficulty altogether, by solving directly for rectangular coordinates and velocities. These considerations apply both to papers by the author and by Izsak on the original spheroidal method and to the author's later papers incorporating the third zonal harmonic into the spheroidal potential.

The present paper simplifies orbital calculations by the spheroidal method for satellite orbits with all inclinations. Its main points are the bypassing of the right ascension and the avoidance of differences of almost equal quantities, so that all calculations become well-conditioned.

1. INTRODUCTION

Objections have sometimes been made to applying the author's spheroidal method to calculate a polar orbit of an artificial satellite. The coordinates that appear are ρ, for which the level surfaces are oblate spheroids, η, for which they are hyperboloids of one sheet, and the right ascension ϕ. The apparent difficulty in a polar orbit arises only in ϕ and then only at a pole.

For an exactly polar orbit I have already shown by limiting processes in V1961a and V1961b[1] that the spheroidal potential leads to ϕ = constant, except at a pole, where it jumps by $\pm 180°$, accordingly as we call the orbit direct or retrograde, respectively. This is the expected behavior, just the same as for a Keplerian orbit, so that no real difficulty appears. It holds whether or not the model takes into account the third zonal harmonic, with coefficient J_3.

Although the difficulty was easily disposed of, without tedious numerical calculations, for an exactly polar orbit, one might still claim that it remains troublesome for an almost polar orbit. For such an orbit the calculation of ϕ involves a small denominator which almost vanishes at a pole. One then may very likely lose accuracy in passing by the pole or have to use special procedures which will increase computer time and storage demands and which will not elsewhere be necessary. The present paper shows how to avoid such difficulties.

2. THE AUTHOR'S SPHEROIDAL SOLUTION; WITHOUT J_3

The notation in this section is that of V1961a, corrections of which are to be found in Walden and Watson 1967, p. 16. The rectangular coordinates X, Y, Z satisfy

$$X + iY = (\rho^2+c^2)^{1/2}(1-\eta^2)^{1/2} \exp i\phi \tag{1.1}$$

$$Z = \rho\eta \tag{1.2}$$

Now by (8.50) of V1961a,

$$\phi = \Omega' + F , \tag{2}$$

1. The initial V refers to the author's own papers.

where F is that part of the expression which varies rapidly near a pole. Here Ω' is given by Eq. (9) of the present paper and

$$F = K\chi \tag{3.1}$$

$$K = |K| \ \mathrm{sgn}\ \alpha_3 \ , \tag{3.2}$$

where

$$K^2 = \alpha_3{}^2\eta_0{}^2\eta_2{}^2(\alpha_2{}^2-\alpha_3{}^2)^{-1}(\eta_0{}^2+\eta_2{}^2-1-\eta_0{}^2\eta_2{}^2)^{-1} \tag{4}$$

But

$$\eta_0{}^2+\eta_2{}^2 = 1+\alpha_2{}^2(-2\alpha_1 c^2)^{-1} \qquad \text{(4.1 of V 1961a)}$$

$$\eta_0{}^2\eta_2{}^2 = (\alpha_2{}^2-\alpha_3{}^2)(-2\alpha_1 c^2)^{-1} \qquad \text{(4.2 of V 1961a)}$$

It follows that $K^2 = 1$, so that

$$K = \mathrm{sgn}\ \alpha_3 = \pm 1 \tag{5}$$

for direct of retrograde orbits, respectively, in order that the right ascension ϕ may correspondingly either increase or always decrease. Then

$$\phi = \Omega' + \chi \ \mathrm{sgn}\ \alpha_3 \tag{6}$$

is an exact equation for all orbits, with the spheroidal model. This is in contradistinction to the results of V1961a, where it was only shown to hold for polar orbits. Thus the present work simplifies all calculations with the spheroidal model.

To find the rectangular coordinates X and Y directly, without first calculating ϕ, insert (6) into (1.1), use

$$\exp i\chi = (1-\eta_0{}^2 \sin^2 \psi)^{-\frac{1}{2}} (\cos \psi + i\sqrt{1-\eta_0{}^2} \sin \psi) \tag{7}$$

from the last paragraph of V1961b, and then put $\eta=\eta_0 \sin \psi$ and $(1-\eta_0{}^2)^{1/2} = |\cos I|$, from (6.4) and (4.7) of V1961a. The troublesome denominator $(1-\eta^2)^{1/2}$ then cancels out, with the result

$$X+iY = (\rho^2+c^2)^{\frac{1}{2}} (\cos\psi + i \cos I \sin \psi) \exp i\Omega' \tag{8}$$

for all orbits, direct or retrograde. Here

$$\Omega' = \beta_3+\alpha_3(\alpha_2{}^2-\alpha_3{}^2)^{-\frac{1}{2}} \eta_0(B_3\psi+ \frac{3}{32} \eta_0{}^2\eta_2^{-4} \sin 2\psi)$$

$$-c^2\alpha_3(-2\alpha_1)^{-1/2}(A_3v+ \sum_{k=1}^{4} A_{3k}\sin kv) \tag{9}$$

Separately

$$X=(\rho^2+c^2)^{\frac{1}{2}} (\cos \Omega' \cos \psi-\sin \Omega' \cos I \sin \psi) \tag{10.1}$$

$$Y=(\rho^2+c^2)^{\frac{1}{2}} (\sin \Omega' \cos \psi + \cos \Omega' \cos I \sin \psi) \tag{10.2}$$

These expression contain no singularities or rapidly varying quantities, so that there is thus never any difficulty with a polar or almost polar orbit. For a strictly polar orbit $\cos I$ and α_3 both vanish, so that $\Omega'=\beta_3$ and

$$X+iY = (\rho^2+c^2)^{1/2} \cos \psi \exp i\beta_3 \tag{11}$$

3. Izsak's Spheroidal Solution

Although Izsak (1960, 1963) suggested using a slowly rotating reference plane to avoid the polar difficulty, actually the same transformations hold for his solution of the spheroidal problem. For the sake of accessibility, I shall refer to his 1963 paper. In making the comparison, note that my symbols are to be changed as follows: $\phi \to \alpha$, $\eta \to \sigma$, $\eta_0 \to s$, and $\beta_3 \to \Omega_*$; others remain the same. Then, with use of Izsak's Eqs. (3), (91), (37), and (63), one finds again the equivalent of the present Eqs. (10) for the rectangular coordinates X and Y. Note that Izsak's expression for Ω' contains $(1-s^2)^{1/2}$ in the numerator and $1-e^2$ in the denominator of each term except Ω_*. The $1-e^2$ in such a denominator does not necessarily produce a singularity as $e \to 1$, since each $(1-e^2)^{-1}$ is multiplied by $\nu=c/a$ and $p = a(1-e^2)$ is a quantity analogous to the semi-latus rectum in a Keplerian orbit. In such an orbit

$p > 0$ for any orbit that does not intersect the center of the planet, even if $e=1$. Incidentally, the same powers of p occur in coefficients B_3, A_3 and the A_{3k}'s.

4. Isolation of the Right Ascension

In either solution, the quantity here called χ is the sensitive part of the expression for the right ascension ϕ. If one actually wants values of ϕ near a pole in an almost polar orbit, it is better to rewrite Eq. (7) as

$$\exp i\chi = (\cos^2\psi + \cos^2 I \sin^2\psi)^{-\frac{1}{2}} (\cos\psi + i|\cos I|\sin\psi) \tag{12}$$

One thus avoids calculating the difference of two almost equal numbers in the denominator. Then ϕ is given by (6) and (12).

5. Velocity Components, with $J_3=0$

On taking the logarithmic derivative of (8) and multiplying the result by X+iY, we find

$$\dot{X}+i\dot{Y} = \left(\frac{\rho\dot{\rho}}{\rho^2+c^2} + i\dot{\Omega}' \right)(X+iY) + (\rho^2+c^2)^{\frac{1}{2}} (-\sin\psi + i \cos I \cos \psi)\dot{\psi}\varepsilon^{i\Omega'}, \tag{13}$$

so that

$$\dot{X} = \frac{\rho\dot{\rho}}{\rho^2+c^2} X - Y\dot{\Omega}' + (\rho^2+c^2)^{\frac{1}{2}} (-\sin\psi \cos\Omega' - \cos I \cos\psi \sin\Omega')\dot{\psi} \tag{14.1}$$

$$\dot{Y} = \frac{\rho\dot{\rho}}{\rho^2+c^2} Y + \chi\dot{\Omega}' + (\rho^2+c^2)^{\frac{1}{2}} (-\sin\psi \sin\Omega' + \cos I \cos\psi \cos\Omega')\dot{\psi} \tag{14.2}$$

Differentiation of (1.2) gives

$$\dot{Z} = \eta\dot{\rho} + \rho\dot{\eta} = \eta\dot{\rho} + \eta_0 \rho \cos\psi \, \dot{\psi} \tag{15}$$

These equations contain neither small denominators nor differences of almost equal quantities. Here

$$\dot{\rho} = ae\left(\frac{\mu}{a_0}\right)^{\frac{1}{2}} (\rho^2+A\rho+B)^{\frac{1}{2}} (\rho^2+c^2\eta^2)^{-1} \sin E \tag{16}$$

from p. 6 of Bonavito 1962, and

$$\dot{\eta} = \eta_0 \cos\psi \; \dot{\psi} = (\alpha_2{}^2-\alpha_3{}^2)^{\frac{1}{2}} (1-q^2\sin^2\psi)^{\frac{1}{2}} (\rho^2+c^2\eta^2)^{-1} \cos \psi, \tag{17}$$

from p. 15 of Walden 1967, after a few transformations. Here $q=\eta_0/\eta_2$. Then

$$\dot{\psi} = \eta_0^{-1} (\alpha_2{}^2-\alpha_3{}^2)^{\frac{1}{2}} (\rho^2+c^2\eta^2)^{-1} (1-q^2\sin^2\psi)^{\frac{1}{2}} \tag{18}$$

Finally, by Eq. (9) of the present paper,

$$\dot{\Omega}' = \alpha_3 (\alpha_2{}^2-\alpha_3{}^2)^{-\frac{1}{2}} \eta_0 (B_3 + \frac{3}{16} \eta_0{}^2 \eta_2^{-4} \cos 2\psi) \dot{\psi}$$
$$-c^2 \alpha_3 (-2\alpha_1)^{-\frac{1}{2}} (A_3 + \sum_{k=1}^{4} kA_{3k} \cos kv) \dot{v} \tag{19}$$

Thus we also need $\dot{v}$. With

$$\rho = (1+e \cos v)^{-1} p, \tag{20}$$

from (5.12) of V1961a, where $p=a(1-e^2)$, we find

$$\dot{\rho} = \frac{e}{p} \rho^2 \sin v \; \dot{v} \tag{21}$$

Comparison of (16) and (21), with use of the anomaly connection

$$\sin E = \frac{\rho}{p} (1-e^2)^{\frac{1}{2}} \sin v \tag{22}$$

then gives

$$\dot{v} = \frac{a}{\rho} \left[\frac{\mu(1-e^2)}{a_0} \right]^{\frac{1}{2}} \frac{(\rho^2+A\rho+B)^{\frac{1}{2}}}{\rho^2+c^2\eta^2} \tag{23}$$

Eqs. (14), (15), (16), (18), (19), and (23) then give the complete algorithm for finding the velocity components in the spheroidal model, when J_3 is not included.

6. The Author's Spheroidal Solution, with J_3

The notation in this section is that of V1966, corrections of which are to be found in Walden and Watson 1967, pp. 19, 20, 22, 27, and 31. With this solution

$$\phi = \Omega'+G \operatorname{sgn} \alpha_3, \tag{24}$$

where Ω' is given in Eq. (41.4) of the present paper and where G is given by Eq. (150) of V1966, viz

$$G = |\alpha_3|\alpha_2^{-1}u^{\frac{1}{2}}(1-S)^{-\frac{1}{2}}[(h_1+h_2)\ \chi_0+(h_1-h_2)\chi_1] \tag{25}$$

From Eq. (158) of V1966, we have

$$(h_1+h_2)\chi_0+(h_1-h_2)\chi_1=2^{-1}(1-C_2)^{\frac{1}{2}}[(1-C_2)^2-C_1^{\ 2}]^{-\frac{1}{2}}(E_2'+E_3') \tag{26}$$

If u is a solution of the cubic equation (27) of V1966, then by (32.1) and (32.2) of that paper

$$C_2 = \frac{c^2u}{a_0p_0}\ (16),\ C_1=2u\delta p_0^{-1}(1-C_2S)^{-1}(1-C_2), \tag{27}$$

so that

$$\frac{(1-C_2)^2-C_1^{\ 2}}{1-C_2} = \frac{u}{1-S}\left[\left(\frac{1}{u} - \frac{c^2}{a_0p_0}\right)(1-S)-R\right], \tag{28.1}$$

where

$$R \equiv \left(\frac{1}{u} - \frac{c^2}{a_0p_0}\ S\right)^{-2}\left(\frac{2\delta}{p_0}\right)^2(1-S)\left(\frac{1}{u} - \frac{c^2}{a_0p_0}\right) \tag{28.2}$$

By (27) of V1966, however,

$$R = \frac{1}{u} -1 - \frac{c^2}{a_0p_0}\ (1-S) \tag{29}$$

Insertion of (29) into (28) then shows that

$$(1-c_2)^{-1}[(1-c_2)^2-c_1{}^2] = (1-S)^{-1}(u-S) \quad , \tag{30}$$

which, with (26), gives

$$(h_1+h_2)\chi_0+(h_1-h_2)\chi_1 = 2^{-1}(u-S)^{-\frac{1}{2}}(1-S)^{\frac{1}{2}}(E_2{}'+E_3{}') \tag{31}$$

Now, by Eqs. (21.2), (18), and (26) of V1966, for all orbits, direct or retrograde,

$$|\alpha_3|\alpha_2^{-1}u^{\frac{1}{2}} = (u-S)^{\frac{1}{2}} \tag{32}$$

Then, from (25), (31), and (32),

$$G = \frac{1}{2}(E_2{}'+E_3{}') \tag{33}$$

for all orbits, polar or not, and direct or retrograde. This is the same as the expression given in Eqs.(159) of V1966 for the sensitive part of ϕ in the case of a polar orbit. Here, however, we have shown that it holds for all orbits.

To evaluate G, place $E_2{}'= E_2{}'(\psi+\pi/2)$ and $E_3{}'=E_3{}'(\psi-\frac{\pi}{2})$ into Eqs.(104) of V1966. The results are

$$\begin{aligned} \cos E_2{}' &= \frac{e_2-\sin\psi}{1-e_2\sin\psi} & \cos E_3{}' &= \frac{e_3+\sin\psi}{1+e_3\sin\psi} \\ \sin E_2{}' &= \frac{(1-e_2{}^2)^{\frac{1}{2}}\cos\psi}{1-e_2\sin\psi} & \sin E_3{}' &= -\frac{(1-e_3{}^2)^{\frac{1}{2}}\cos\psi}{1+e_3\sin\psi} \quad , \end{aligned} \tag{34}$$

where

$$e_2=(1-P)^{-1}Q, \quad e_3=(1+P)^{-1}Q \ , \quad Q^2=P^2+S \ , \tag{35}$$

with $0 \leqq e_3 \leqq e_2 \leqq 1$, by Eqs. (100) and (47) of V1966. Then, by (33),

$$\cos(E_2{}'+E_3{}') = \cos 2G = 2\cos^2 G-1 \tag{36}$$

From (34) and (36) it then follows that

$$\cos G = k(\psi)\ \frac{2^{-\frac{1}{2}}\left(1+e_2e_3 + \sqrt{(1-e_2{}^2)(1-e_3{}^2)}\right)^{\frac{1}{2}}\cos\psi}{[(1-e_2\sin\psi)(1+e_3\sin\psi)]^{\frac{1}{2}}}\ , \tag{37}$$

where $k(\psi)=\pm 1$.

We now show that $k(\psi)=1$ for all ψ. First note that $E_2'(y)$ is related to y in the same way that an eccentric anomaly is related to a true anomaly. The same holds for $E_3'(y)$. Thus each increases as y increases, by Eq. (160) of V1966, so that $G\equiv 2^{-1}\times [E_2'(\psi+\pi/2)+E_3'(\psi-\pi/2)]$ is a continuous monotonically increasing function of ψ.

Also, from the definitions, $E_2'(y)$ and $E_3'(y)$ are both equal to $n\pi$ for $y = n\pi$. Thus

$$G = \psi \quad \text{for } \psi = (n+\tfrac{1}{2})\pi,\ (n=0,1,2\ldots) \tag{38}$$

so that $\cos\psi$ and $\cos G$ both vanish for $\psi=(n+\frac{1}{2})\pi$. Now consider a small interval $(n+\frac{1}{2})\pi-\epsilon \leqq \psi \leqq (n+\frac{1}{2})\pi + \epsilon$. Since G always increases with increase in ψ, the corresponding changes $\Delta\cos\psi$ and $\Delta\cos G$ are both negative if n is even and both positive if n is odd. Thus $k(\psi) > 0$ over any such inteval. But $k(\psi) = \pm 1$ for all ψ and since $\cos G$ and thus $k(\psi)$ are continuous functions of ψ, it follows that

$$k(\psi) = 1 \text{ for all } \psi \tag{39}$$

Before we rewrite (37) with omission of $k(\psi)$, let us first simplify it. To do so, note that by (35) and by (48) of V1966, which is

$$\eta = P + Q\sin\psi, \tag{40}$$

we obtain

$$(1-e_2\sin\psi)(1+e_3\sin\psi) = (1-P^2)^{-1}(1-\eta^2) \tag{41}$$

Now from Eq. (32.3) of V1966

$$2P = r(1-S)\delta, \tag{42.1}$$

where

$$\delta = \frac{r_e}{2} J_2^{-1} |J_3| \tag{42.2}$$

$$r = 2(1-C_2 S)^{-1} u/p_o \tag{42.3}$$

Thus $\delta = O(J_2)$ and $r = O(1)$. Eqs. (35) and (42) then show that

$$1+e_2 e_3 + \sqrt{(1-e_2^2)(1-e_3^2)} = (1-P^2)^{-1} (1+S+(1-S)\sqrt{1-r^2\delta^2}\ ) \tag{43}$$

On inserting (39), (41), and (43) into (37), we find

$$\cos G = 2^{-\frac{1}{2}} (1-\eta^2)^{-\frac{1}{2}} [1+S+(1-S)\sqrt{1-r^2\delta^2}\ ]^{\frac{1}{2}} \cos\psi \tag{44}$$

We also need sin G in calculating rectangular coordinates. To evaluate it unambiguously first note that

$$2 \sin G \cos G = \sin (E_2{}'+E_3{}') \tag{45}$$

$$=(1-\eta^2)^{-1}(1-P^2)[(e_3\sqrt{1-e_2^2} - e_2\sqrt{1-e_3^2}) + (\sqrt{1-e_2^2} + \sqrt{1-e_3^2}\ )\sin\psi]\cos\psi \tag{46}$$

by (24) and (31). Then from (35), (42), (44), (45), and (46) it follows that

$$\sin G = \frac{2^{-\frac{1}{2}}(1-S)^{\frac{1}{2}}}{(1-\eta^2)^{\frac{1}{2}}} \frac{\{Q(\sqrt{1-r\delta} - \sqrt{1+r\delta}) + [(1+P)\sqrt{1-r\delta} + (1-P)\sqrt{1+r\delta}]\sin\psi\}}{[1+S+(1-S)\sqrt{1-r^2\delta^2}\ ]^{\frac{1}{2}}} \tag{47}$$

To check this, note that for $J_3 = 0$ we have $\delta = 0$, $P = 0$, $Q = S^{1/2}$, and $S = \sin^2 I$, so that (47) then reduces to

$$\sin G = (1-\eta^2)^{-\frac{1}{2}} |\cos I| \sin\psi, \tag{48}$$

agreeing with (7) for $\sin\chi$.

If one really wants values of the right ascension near a pole, one can use (24), (44), and (47).

It is then advisable, however, to rewrite the $1-\eta^2$ in the denominator by using (35) and (40). One finds

$$1-\eta^2 = \cos^2\psi-(P^2+2PQ\ \sin\ \psi)+(1-S-P^2)\sin^2\psi, \tag{49}$$

resulting in the same kind of simplification near a pole as does (12).

Near a pole in a nearly polar orbit the term $-(P^2+2PQ\ \sin\psi)$ in (49) is much smaller than the positive term $(1-S-P^2)\sin^2\psi$. To verify this statement, note that in a nearly polar orbit, $S\approx 1$, $Q\approx 1$, $P<<1$, and near a pole $|\sin\ \psi|\approx 1$. Then from (32.3) of V1966

$$P=(1\ -\ \frac{c^2}{a_0p_0}\ Su)^{-1}\ \frac{\delta}{p_o}\ u(1-S)\ \approx\frac{7}{6400}\ (1-S), \tag{49.1}$$

so that

$$|P^2\ +\ 2PQ\ \sin\ \psi|\ \approx\frac{7}{3200}\ (1-S) \tag{49.2}$$

and

$$(1-S-P^2)\sin^2\psi\ \approx\ 1-S \tag{49.3}$$

Thus Eq. (49) gives no trouble near a pole.

In rectangular coordinates we find from (1),(24),(44), and (47)

$$X\ =\ (\rho^2+c^2)^{\frac{1}{2}}[H_1\cos\ \Omega'\ \cos\ \psi\ -H_1^{-1}\sqrt{1-S}\ \operatorname{sgn}\alpha_3\sin\Omega'(H_2+H_3\sin\psi)] \tag{50.1}$$

$$Y\ =\ (\rho^2+c^2)^{\frac{1}{2}}[H_1\sin\ \Omega'\ \cos\ \psi\ +H_1^{-1}\sqrt{1-S}\ \operatorname{sgn}\alpha_3\cos\Omega'(H_2+H_3\sin\psi)] \tag{50.2}$$

and

$$Z\ =\ \rho\eta-\delta\ , \tag{50.3}$$

from (1.2) of V1966. Here

$$H_1\ =\ 2^{-\frac{1}{2}}[1\ +\ S\ +\ (1-S)\ \sqrt{1-r^2\delta^2}\ ]^{\frac{1}{2}} \tag{51.1}$$

$$H_2\ =\ \frac{1}{2}\ Q(\ \sqrt{1-r\delta}\ -\ \sqrt{1-r\delta}\ ) \tag{51.2}$$

$$H_3\ =\ \frac{1}{2}[\ (1+P)\ \sqrt{1-r\delta}\ +\ (1-P)\sqrt{1\ +\ r\delta}\ ] \tag{51.3}$$

and

$$\Omega'=\beta_3-c^2\alpha_3(-2\alpha_1)^{-\frac{1}{2}}(A_3v+\sum_{k=1}^{4}A_{3k}\sin kv)$$

$$+\alpha_3\alpha_2^{-1}u^{\frac{1}{2}}(B_3\psi-\frac{3}{4}C_1C_2Q\cos\psi+\frac{3}{32}C_2^{\,2}Q^2\sin 2\psi), \tag{51.4}$$

from Eq.(150) of V1966. Like Eqs.(10) these equations contain no singularities, even for a polar orbit. Moreover they hold for all oribts.

For an exactly polar orbit we have S=1, P=0, Q=1, α_3=0, and $\Omega'=\beta_3$. The X and Y equations then become

$$X+iY=(\rho^2+c^2)^{\frac{1}{2}}\cos\psi\exp i\beta_3, \tag{52}$$

as for the case J_3=0 of Eq. (11). The Z equation, however, is $Z=\rho\eta-\delta$, where $\delta=(r_e/2)J_2^{-1}|J_3|$, so that the orbit is still changed by the J_3.

7. Velocity Components, with J_3 Accounted for

From Eqs. (50.1) and (50.2)

$$X+iY=(\rho^2+c^2)^{\frac{1}{2}}[H_1\cos\psi+iH_1^{-1}(1-S)^{\frac{1}{2}}\operatorname{sgn}\alpha_3(H_2+H_3\sin\psi)]\exp i\Omega' \tag{53}$$

Logarithmic differentiation of (53), with multiplication of the result by X+iY, gives

$$\dot X+i\dot Y=(\frac{\rho\dot\rho}{\rho^2+c^2}+i\dot\Omega')(X+iY)+(\rho^2+c^2)^{\frac{1}{2}}[-H_1\sin\psi+iH_1^{-1}(1-S)^{\frac{1}{2}}\operatorname{sgn}\alpha_3H_3\cos\psi]\dot\psi\exp i\Omega', \tag{54}$$

so that

$$\dot X=\frac{\rho\dot\rho}{\rho^2+c^2}X-Y\dot\Omega'+(\rho^2+c^2)^{\frac{1}{2}}[-H_1\sin\psi\cos\Omega'-H_1^{-1}(1-S)^{\frac{1}{2}}\operatorname{sgn}\alpha_3H_3\cos\psi\sin\Omega']\dot\psi \tag{54.1}$$

$$\dot{Y}= -\frac{\rho\dot{\rho}}{\rho^2+c^2}\,Y+X\dot{\Omega}'+(\rho^2+c^2)^{\frac{1}{2}}[-H_1\sin\psi\sin\Omega'+H_1^{-1}(1-S)^{\frac{1}{2}}\mathrm{sgn}\,\alpha_3H_3\cos\psi\cos\Omega']\dot{\psi} \tag{54.2}$$

Also

$$\dot{Z} = \eta\dot{\rho} + \rho\dot{\eta} \tag{54.3}$$

by (15). Eqs. (16) and (23) still hold, so that the equations for $\dot{\rho}$ and $\dot{v}$ are as for $J_3=0$.

For $\dot{\Omega}'$ we find from (51.4)

$$\dot{\Omega}'=-c^2\alpha_3(-2\alpha_1)^{-\frac{1}{2}}(A_3+\sum_{k=1}^{4}kA_{3k}\cos kv)\dot{v}$$

$$+\alpha_3\alpha_2^{-1}u^{\frac{1}{2}}(B_3+\tfrac{3}{4}C_1C_2Q\sin\psi+\tfrac{3}{16}\,C_2{}^2Q^2\cos 2\psi)\dot{\psi} \tag{55}$$

The new expression for $\dot{\psi}$ is still lacking. From p. 14 of Bonavito 1966, we find

$$\eta = Q\cos\psi \qquad \dot{\psi} = \frac{Q}{\rho^2+c^2\eta^2}\left(\frac{\mu p_0}{u}\right)^{\frac{1}{2}}(1+C_1\eta-C_2\eta^2)^{\frac{1}{2}}\cos\psi\ , \tag{56}$$

so that

$$\dot{\psi} = \left(\frac{\mu p_0}{u}\right)^{\frac{1}{2}}\frac{(1+C_1\eta-C_2\eta^2)^{\frac{1}{2}}}{\rho^2+c^2\eta^2} \tag{57}$$

Here

$$\frac{1}{u} = 1+\frac{c^2}{a_0p_0}\,(1-S)+\frac{\left(\frac{2\delta}{p_0}\right)^2(1-S)(1-\frac{c^2}{a_0p_0}\,S)}{[1+\frac{c^2}{a_0p_0}\,(1-2S)]^2}+O(J_2{}^4), \tag{58}$$

$$c^2 = c_0{}^2-\delta^2 = r_e{}^2J_2 - \frac{1}{4}\,r_e{}^2J_3{}^2J_2^{-2} \tag{59}$$

The equations of this section reduce to those of Section 5, if J_3 is equated to zero.

8. The Improved Algorithm for the Spheroidal Model, with J_3

Begin with Section 12 of V1966 and follow it through the third line on p.45, viz, $\eta = P+Q \sin \psi$. Instead of then calculating E_2' and E_3', however, replace that calculation with Eqs. (42), (50), and (51) of the present paper. This changed procedure not only simplifies the calculation of X, Y, and Z for near-polar orbits but bypasses the right ascension in all cases. To calculate the velocities $\dot{X}$, $\dot{Y}$, and $\dot{Z}$, use Eqs. (54) through (59) of the present paper.

9. References

Bonavito, N. L., 1962, NASA Technical Note D-1177

Bonavito, N. L., 1966, NASA Technical Note D-3562

Izsak, I., 1960, Smithsonian Institution Astrophysical Observatory Research in Space Science, Special Report No. 52.

Izsak, I., 1963, Smithsonian Contributions to Astrophysics, Vol. 6, Research in Space Science, 81-107

Vinti, J. P., 1959, J. Research Nat. Bureau Standards, 63B, 105-116

Vinti, J. P., 1961a, J. Research Nat. Bureau Standards, 65B, 169-201

Vinti, J. P., 1961b, Astron. J., 66, 514-516

Vinti, J. P., 1966, J. Research Nat. Bureau Standards, 70B, 17-46

Walden, H. and Watson, S., 1967, NASA Technical Note TN-D-4088

Walden, H., 1967, NASA Technical Note TN D-3803

PRECEDING PAGE BLANK NOT FILMED.

Appendix I

Algorithm for Satellite Position Vector and Velocity, if the Potential $V = -\mu\rho(\rho^2+c^2\eta^2)^{-1}(J_3 = 0)$

Given

$$\mu,\ r_e,\ J_2\ a,\ e,\ I,\ \beta_1,\ \beta_2,\ \beta_3$$

Compute once for each orbit:

$$c_0{}^2=r_e{}^2J_2,\quad \eta_0=\sin I,\ p=a(1-e^2),\ D=(ap-c_0{}^2)(ap-c_0{}^2\eta_0{}^2)+4a^2c_0{}^2\eta_0{}^2$$

$$D'=D+4a^2c_0{}^2(1-\eta_0{}^2),\ A=-2ac_0{}^2D^{-1}(1-\eta_0{}^2)(ap-c_0{}^2\eta_0{}^2)<0,\ B=c_0{}^2\eta_0{}^2D^{-1}D'$$

$$b_1=-\tfrac{1}{2}A>0,\ b_2=B^{\frac{1}{2}},\ a_0=a+b_1>a,\ p_0=-c_0{}^2a_0^{-1}(1-\eta_0^{-1})+aa_0^{-1}pD^{-1}D'$$

$$\alpha_2=(\mu p_0)^{\frac{1}{2}},\ \alpha_3=\alpha_2\left(1-\frac{c_0{}^2\eta_0{}^2}{a_0p_0}\right)\cos I,\ \eta_2^{-2}=\frac{c_0{}^2D}{apD'},\ q=\eta_0\eta_2^{-1}$$

$$\alpha_2' = \alpha_2(1+\frac{c_0{}^2}{a_0p_0}\ \cos^2 I)^{\frac{1}{2}}$$

Also, with $R_n(x) \equiv x^nP_n(x^{-1})$,

compute

$$A_1=(1-e^2)^{\frac{1}{2}}p\sum_{n=2}^{\infty}\left(\frac{b_2}{p}\right)^n P_n\left(\frac{b_1}{b_2}\right)R_{n-2}[(1-e^2)^{\frac{1}{2}}]$$

$$A_2=(1-e^2)^{\frac{1}{2}}p^{-1}\sum_{n=0}^{\infty}(b_2/p)^nP_n(b_1/b_2)R_n[(1-e^2)^{\frac{1}{2}}],\ \text{where}$$

$$D_{2i}=\sum_{n=0}^{i}(-1)^{i-n}(c_0/p)^{2i-2n}(b_2/p)^{2n}P_{2n}(b_1/b_2)$$

$$D_{2i+1}=\sum_{n=0}^{i}(-1)^{i-n}(c_0/p)^{2i-2n}(b_2/p)^{2n+1}P_{2n+1}(b_1/b_2)$$

$$A_3=(1-e^2)^{\frac{1}{2}}p^{-3}\sum_{m=0}^{\infty}D_m R_{m+2}[(1-e^2)^{\frac{1}{2}}]$$

$$B_1 = 2\pi^{-1}q^{-2}[K(q)-E(q)]= \frac{1}{2} + \frac{3}{16}q^2+ \frac{15}{128}q^4+ \frac{175}{2048}q^6+\ldots\ldots$$

$$B_2 = 2\pi^{-1}K(q) = 1+ \frac{1}{4}q^2+ \frac{9}{64}q^4+ \frac{25}{256}q^6+\ldots.$$

$$a_0' = a_0+A_1+c_0{}^2\eta_0{}^2A_2B_1B_2^{-1}$$

$$\gamma_m = \frac{(2m)!}{2^{2m}(m!)^2}\sum_{n=1}^{m-1}\frac{(2n)!\,\eta_0^{2n}}{2^{2n}(n!)^2}$$

$$B_3 =1-(1- \eta_2^{-2})^{-\frac{1}{2}} - \sum_{m=2}^{\infty}\gamma_m\eta_2^{-2m}$$

$$A_{11} = \frac{3}{4}(1-e^2)^{\frac{1}{2}}p^{-3}e(-2b_1b_2{}^2p+b_2{}^4) \qquad A_{12}= \frac{3}{32}p^{-3}(1-e^2)^{\frac{1}{2}}b_2{}^4e^2$$

$$A_{21}=(1-e^2)^{\frac{1}{2}}p^{-1}e[b_1p^{-1}+(3b_1{}^2-b_2{}^2)p^{-2}- \frac{9}{2}b_1b_2{}^2(1+\frac{e^2}{4})p^{-3}+ \frac{3}{8}b_2{}^4(4+3e^2)p^{-4}]$$

$$A_{22}=(1-e^2)^{\frac{1}{2}}p^{-1}[\frac{e^2}{8}(3b_1{}^2-b_2{}^2)p^{-2}- \frac{9}{8}e^2b_1b_2{}^2p^{-3}+ \frac{3}{32}b_2{}^4(6e^2+e^4)p^{-4}]$$

$$A_{23}=(1-e^2)^{\frac{1}{2}}p^{-1}\frac{e^3}{8}(-b_1b_2{}^2p^{-3}+b_2{}^4p^{-4}),\quad A_{24}= \frac{3}{256}(1-e^2)^{\frac{1}{2}}p^{-5}b_2{}^4e^4$$

$$A_{31}=(1-e^2)^{\frac{1}{2}}p^{-3}e[2+b_1p^{-1}(3+ \frac{3}{4}e^2)-p^{-2}(\frac{1}{2}b_2{}^2+c_0{}^2)(4+3e^2)]$$

$$A_{32}=(1-e^2)^{\frac{1}{2}}p^{-3}[\frac{e^2}{4}+\frac{3}{4}b_1p^{-1}e^2-p^{-2}(\frac{e^4}{4}+\frac{3}{2}e^2)(\frac{1}{2}b_2{}^2+c_0{}^2)]$$

$$A_{33}=(1-e^2)^{\frac{1}{2}}p^{-3}e^3[\frac{1}{12}b_1p^{-1}-\frac{1}{3}p^{-2}(\frac{1}{2}b_2{}^2+c_0{}^2)],\quad A_{34}=-\frac{1}{32}(1-e^2)^{\frac{1}{2}}p^{-5}e^4(\frac{1}{2}b_2{}^2+c_0{}^2)$$

$$2\pi\nu_1=a_0'^{-1}\left(\frac{\mu}{a_0}\right)^{\frac{1}{2}},\qquad 2\pi\nu_2=a_0'^{-1}\alpha_2'A_2B_2^{-1},\qquad e'=a_0^{-1}ae<e$$

$$\lambda_1=\beta_1-c_0{}^2\beta_2\alpha_2^{-1}\eta_0{}^2B_1B_2^{-1},\quad \lambda_2=\beta_1+\beta_2\alpha_2^{-1}(a_0+A_1)A_2{}^{-1}$$

$$\lambda_3=\left(\frac{\mu}{a_0}\right)^{-\frac{1}{2}}\alpha_2'A_2B_2^{-1},\qquad \lambda_4=a_0{}^{-1}(A_1+c_0{}^2\eta_0{}^2A_2B_1B_2{}^{-1})$$

$$\lambda_5=c_0{}^2\left(\frac{\mu}{a_0}\right)^{\frac{1}{2}}\alpha_2'\eta_0{}^4,\qquad \lambda_6=\left(\frac{\mu}{a_0}\right)^{-\frac{1}{2}}\alpha_2'B_2^{-1},\qquad \lambda_7=\frac{1}{8}q^2B_2^{-1}$$

For each point at time t, now compute

1) $M_s=2\pi\nu_1(t+\lambda_1)$ $\quad\psi_s=2\pi\nu_2(t+\lambda_2)$

2) Solve for E_0: $M_s+E_0-e'\sin(M_s+E_0)=M_s$

3) To find v_0: Place $E=M_s+E_0$ in the anomaly connections

$$\cos v=(1-e\cos E)^{-1}(\cos E-e),\ \sin v=(1-e\cos E)^{-1}(1-e^2)^{\frac{1}{2}}\sin E$$

and solve for $v=M_s+v_0$

4) $\psi_0=\lambda_3v_0$

5) Compute $M_1=-\lambda_4v_0+\frac{1}{4}\lambda_5\sin(2\psi_s+2\psi_0)$

6) Then $E_1=[1-e'\cos(M_s+E_0)]^{-1}M_1-\frac{1}{2}e'[1-e'\cos(M_s+E_0)]^{-3}M_1{}^2\sin(M_s+E_0)$

7) Place $E=M_s+E_0+E_1$ in the anomaly connections and solve for $v=M_s+v_0+v_1$

8) Then $\psi_1=\lambda_6[A_2v_1+\sum_{k=1}^{2}A_{2k}\sin(kM_s+kv_0)]+\lambda_7\sin(2\psi_s+2\psi_0)$

9) Compute

$$M_2=-a_0^{-1}[A_1v_1+\sum_{k=1}^{2}A_{1k}\sin(kM_s+kv_0)+\lambda_5\Big\{B_1\psi_1-\frac{1}{2}\psi_1\cos(2\psi_s+2\psi_0)-\frac{q^2}{8}\sin(2\psi_s+2\psi_0)$$

$$+\frac{q^4}{64}\sin(4\psi_s+4\psi_0)\Big\}]$$

10) Then $E_2=[1-e'\cos(M_s+E_0+E_1)]^{-1}M_2$

11) Place $E=M_s+E_0+E_1+E_2$ in the anomaly connections to find $v=M_s+v_0+v_1+v_2$.

12) Then $\psi_2=\lambda_6[A_2v_2+A_{21}v_1\cos(M_s+v_0)+2A_{22}v_1\cos(2M_s+2v_0)$

$$+A_{23}\sin(3M_s+3v_0)+A_{24}\sin(4M_s+4v_0)]+2\lambda_7[\psi_1\cos(2\psi_s+2\psi_0)$$

$$+\frac{3q^2}{8}\sin(2\psi_s+2\psi_0)-\frac{3q^2}{64}\sin(4\psi_s+4\psi_0)]$$

Then

$$E=M_s+E_0+E_1+E_2,\quad v=M_s+v_o+v_1+v_2,\quad \psi=\psi_s+\psi_0+\psi_1+\psi_2$$

13) $\rho=a(1-e\cos E)=(1+e\cos v)^{-1}p,\quad \eta=\eta_0\sin\psi$

14) $\Omega'=\beta_3+\alpha_3\alpha_2'^{-1}(B_3\psi+\frac{3}{32}\eta_0{}^2\eta_2{}^{-4}\sin 2\psi)-c^2\alpha_3\left(\frac{\mu}{a_0}\right)^{-\frac{1}{2}}(A_3v+\sum_{k=1}^{4}A_{3k}\sin kv)$

Then the rectangular coordinates are given by

15) $X=(\rho^2+c_0{}^2)^{\frac{1}{2}}(\cos\Omega'\cos\psi-\sin\Omega'\cos I\sin\psi)$

16) $Y=(\rho^2+c_0{}^2)^{\frac{1}{2}}(\sin\Omega'\cos\psi+\cos\Omega'\cos I\sin\psi)$

17) $Z=\rho\eta$

To find the velocity components, compute

18) $$\dot{v}=\frac{a}{\rho}\left[\frac{\mu(1-e^2)}{a_0}\right]^{\frac{1}{2}}\frac{(\rho^2+A\rho+B)^{\frac{1}{2}}}{\rho^2+c_0{}^2\eta^2}$$

19) $$\dot{\rho}=\frac{e}{p}\rho^2\sin v\ \dot{v}$$

20) $$\dot{\psi}=\frac{\alpha_2'(1-q^2\sin^2\psi)^{\frac{1}{2}}}{\rho^2+c_0{}^2\eta^2}$$

21) $$\dot{\Omega}'=\alpha_3\alpha_2'^{-1}(B_3+\frac{3}{16}\eta_0{}^2\eta_2{}^{-4}\cos 2\psi)\dot{\psi}$$

$$-c^2\alpha_3\left(\frac{\mu}{a_0}\right)^{-\frac{1}{2}}(A_3+\sum_{k=1}^{4}kA_{3k}\cos kv)\dot{v}$$

Then

22) $$\dot{X}=\frac{\rho\dot{\rho}}{\rho^2+c_0{}^2}X-Y\dot{\Omega}'+(\rho^2+c_0{}^2)^{\frac{1}{2}}(-\sin\psi\cos\Omega'-\cos I\cos\psi\sin\Omega')\dot{\psi}$$

23) $$\dot{Y}=\frac{\rho\dot{\rho}}{\rho^2+c_0{}^2}Y+X\dot{\Omega}'+(\rho^2+c_0{}^2)^{\frac{1}{2}}(-\sin\psi\sin\Omega'+\cos I\cos\psi\cos\Omega')\dot{\psi}$$

24) $$\dot{Z}=\eta\dot{\rho}+\eta_0\rho\cos\psi\ \dot{\psi}$$

Appendix II

Algorithm for Satellite Position Vector and Velocity, if the Potential $V = -\mu(\rho+\eta\delta)(\rho^2+c^2\eta^2)^{-1}$ $(J_3\neq 0)$

Given $\mu, r_e, J_2, J_3, a, e, S, \beta_1, \beta_2, \beta_3$

Compute <u>once for each orbit</u>

$$c_0{}^2 = r_e{}^2 J_2, \quad \delta = \frac{1}{2} r_e \frac{|J_3|}{J_2}, \quad c^2 = c_0{}^2 - \delta^2, \quad p = a(1-e^2)$$

$$A = \frac{-2ac^2(ap-c^2S)(1-S) + \dfrac{8a^2c^2}{p}\delta^2\left\{1+\dfrac{c^2}{ap}(3S-2)\right\}S(1-S)}{(ap-c^2)(ap-c^2S)+4a^2c^2S+\dfrac{4c^2}{p^2}\delta^2(3ap-4a^2-c^2)S(1-S)}$$

$$B = c^2+(2a)^{-1}(ap-c^2)A, \quad b_1 = -\frac{1}{2}A, \quad a_0 = a+b_1, \quad b_2 = B^{\frac{1}{2}}$$

$p_0 = a_0^{-1}(B+ap-2Aa-c^2), \quad \alpha_2 = (\mu p_0)^{\frac{1}{2}}$, u from

$$u^{-1} = 1+\frac{c^2}{a_0p_0}(1-S)+\frac{\left(\dfrac{2\delta}{p_0}\right)^2(1-S)\left(1-\dfrac{c^2}{a_0p_0}S\right)}{\left[1+\dfrac{c^2}{a_0p_0}(1-2S)\right]^2}, \quad C_2 = \frac{c^2}{a_0p_0}u, \quad \alpha_3 = \pm\alpha_2(1-Su^{-1})^{\frac{1}{2}}$$

+ for direct orbit
− for retrograde

$$C_1 = \left(1-\frac{c^2}{a_0p_0}Su\right)^{-1}\frac{2\delta}{p_0}u\left(1-\frac{c^2}{a_0p_0}u\right), \quad P = \left(1-\frac{c^2}{a_0p_0}Su\right)^{-1}\frac{\delta}{p_0}u(1-S),$$

With $R_n(x) = x^nP_n(x^{-1})$, compute

$$A_1 = (1-e^2)^{\frac{1}{2}}p\sum_{n=2}^{\infty}(b_2/p)^nP_n(b_1/b_2)R_{n-2}[(1-e^2)^{\frac{1}{2}}]$$

$$A_2 = (1-e^2)^{\frac{1}{2}}p^{-1}\sum_{n=0}^{\infty}(b_2/p)^nP_n(b_1/b_2)R_n[(1-e^2)^{\frac{1}{2}}]$$

$$A_3=(1-e^2)^{\frac{1}{2}}p^{-3}\sum_{m=0}^{\infty} D_m R_{m+2}[(1-e^2)^{\frac{1}{2}}], \text{ where}$$

$$D_{2i} = \sum_{n=0}^{i} (-1)^{i-n}(c/p)^{2i-2n}(b_2/p)^{2n}P_{2n}(b_1/b_2)$$

$$D_{2i+1}= \sum_{n=0}^{i} (-1)^{i-n}(c/p)^{2i-2n}(b_2/p)^{2n+1}P_{2n+1}(b_1/b_2)$$

$$A_{11}= \frac{3}{4}(1-e^2)^{\frac{1}{2}}p^{-3}e(-2b_1b_2{}^2p+b_2{}^4) \qquad A_{12}= \frac{3}{32}p^{-3}(1-e^2)^{\frac{1}{2}}b_2{}^4e^2$$

$$A_{21}= (1-e^2)^{\frac{1}{2}}p^{-1}e[b_1p^{-1}+(3b_1{}^2-b_2{}^2)p^{-2}-\frac{9}{2}b_1b_2{}^2(1+\frac{e^2}{4})p^{-3}+\frac{3}{8}b_2{}^4(4+3e^2)p^{-4}]$$

$$A_{22}=(1-e^2)^{\frac{1}{2}}p^{-1}[\frac{e^2}{8}(3b_1{}^2-b_2{}^2)p^{-2}-\frac{9}{8}e^2b_1b_2{}^2p^{-3}+\frac{3}{32}b_2{}^4(6e^2+e^4)p^{-4}]$$

$$A_{23}=(1-e^2)^{\frac{1}{2}}p^{-1}\frac{e^3}{8}(-b_1b_2{}^2p^{-3}+b_2{}^4p^{-4}), \qquad A_{24}= \frac{3}{256}(1-e^2)^{\frac{1}{2}}p^{-5}b_2{}^4e^4$$

$$A_{31}=(1-e^2)^{\frac{1}{2}}p^{-3}e[2+b_1p^{-1}(3+\frac{3}{4}e^2)-p^{-2}(\frac{1}{2}b_2{}^2+c^2)(4+3e^2)]$$

$$A_{32}=(1-e^2)^{\frac{1}{2}}p^{-3}[\frac{e^2}{4}+\frac{3}{4}b_1p^{-1}e^2-p^{-2}(\frac{e^4}{4}+\frac{3}{2}e^2)(\frac{1}{2}b_2{}^2+c^2)]$$

$$A_{33}=(1-e^2)^{\frac{1}{2}}p^{-3}e^3[\frac{1}{12}b_1p^{-1}-\frac{1}{3}p^{-2}(\frac{1}{2}b_2{}^2+c^2)], \qquad A_{34}= -\frac{1}{32}(1-e^2)^{\frac{1}{2}}p^{-5}(\frac{1}{2}b_2{}^2+c^2)e^4$$

$$Q = (P^2+S)^{1/2}$$

$$B_2=1-\frac{1}{2}C_1P+(\frac{3}{8}C_1{}^2+\frac{1}{2}C_2)(\frac{1}{2}Q^2)+\frac{9}{64}C_2{}^2Q^4+0(J_2{}^3)$$

$$B_1'= \frac{1}{2}Q^2+P^2-\frac{3}{4}C_1PQ^2+\frac{3}{64}(4C_2+3C_1{}^2)Q^4+\frac{15}{128}C_2{}^2Q^6+0(J_2{}^3)$$

$$B_3 = -\frac{1}{2}C_2 - \frac{3}{8}C_1^2 - \frac{3}{8}C_2^2\left(1+\frac{1}{2}Q^2\right) + O(J_2^3),$$

$$B_{11} = -2PQ + \frac{3}{8}C_1Q^3, \quad B_{12} = -\left(\frac{Q^2}{4} + \frac{1}{8}C_2Q^4\right), \quad B_{13} = -C_1\frac{Q^3}{24}, \quad B_{14} = C_2\frac{Q^4}{64}$$

$$B_{21} = -C_2PQ + \frac{9}{16}C_1C_2Q^3 + \frac{1}{2}C_1Q, \quad B_{22} = -\frac{1}{32}\left[(4C_2+3C_1^2)Q^2 + 3C_2^2Q^4\right]$$

$$B_{23} = -\frac{1}{16}C_1C_2Q^3, \quad B_{24} = \frac{3}{256}C_2^2Q^4, \quad r = 2(1-C_2S)^{-1}up_0^{-1}$$

$$a_0' = a_0 + A_1 + c^2A_2B_1'B_2^{-1}, \quad 2\pi\nu_1 = \left(\frac{\mu}{a_0}\right)^{\frac{1}{2}}(a_0')^{-1}, \quad 2\pi\nu_2 = \alpha_2u^{-\frac{1}{2}}A_2B_2^{-1}(a_0')^{-1},$$

$$e' = a\,e\,a_0^{-1}$$

$$\lambda_1 = \beta_1 - c^2\beta_2\alpha_2^{-1}B_1'B_2^{-1}, \quad \lambda_2 = \beta_1 + \beta_2\alpha_2^{-1}(a_0+A_1)A_2^{-1}, \quad \lambda_3 = \left(\frac{\mu}{a_0}\right)^{-\frac{1}{2}}\alpha_2u^{-\frac{1}{2}}A_2B_2^{-1}$$

$$\lambda_4 = a_0^{-1}(A_1 + c^2A_2B_1'B_2^{-1}), \quad \lambda_5 = \left(\frac{\mu}{a_0}\right)^{\frac{1}{2}}c^2\alpha_2^{-1}u^{\frac{1}{2}}, \quad \lambda_6 = \left(\frac{\mu}{a_0}\right)^{-\frac{1}{2}}\alpha_2u^{-\frac{1}{2}}B_2^{-1},$$

$$H_1 = 2^{-\frac{1}{2}}\left[1+S+(1-S)(1-r^2\delta^2)^{\frac{1}{2}}\right]^{\frac{1}{2}} \quad H_2 = \frac{1}{2}Q\left[(1-r\delta)^{\frac{1}{2}} - (1+r\delta)^{\frac{1}{2}}\right],$$

$$H_3 = \frac{1}{2}\left[(1+P)(1-r\delta)^{\frac{1}{2}} + (1-P)(1+r\delta)^{\frac{1}{2}}\right]$$

<u>Compute for each point</u>

1) $M_s = 2\pi\nu_1(t+\lambda_1)$, $\quad \psi_s = 2\pi\nu_2(t+\lambda_2)$

2) Solve for E_0: $\quad M_s + E_0 - e'\sin(M_s+E_0) = M_s$

3) To find v_0: Place $E = M_s + E_0$ in the <u>anomaly connections</u>

4) $\psi_0 = \lambda_3 v_0$

5) Compute $M_1 = -\lambda_4 v_0 - a_0^{-1}\lambda_5 B_{12}\sin(2\psi_s + 2\psi_0)$

6) Then $E_1 = [1 - e'\cos(M_s + E_0)]^{-1} M_1 - \frac{1}{2}e'[1 - e'\cos(M_s + E_0)]^{-3} M_1{}^2 \sin(M_s + E_0)$

7) Place $E = M_s + E_0 + E_1$ in the <u>anomaly connections</u> and solve for $v = M_s + v_0 + v_1$

8) Then $\psi_1 = \lambda_6 [A_2 v_1 + \sum_{k=1}^{2} A_{2k}\sin(kM_s + kv_0)] - B_{21}B_2^{-1}\cos(\psi_s + \psi_0) - B_{22}B_2^{-1}\cdot$

$\sin(2\psi_s + 2\psi_0)$

9) Compute $M_2 = -a_0{}^{-1}[A_1 v_1 + \sum_{k=1}^{2} A_{1k}\sin(kM_s + kv_0) + \lambda_5 \{B_1{}'\psi_1 + B_{11}\cos(\psi_s + \psi_0)$

$+ 2B_{12}\psi_1\cos(2\psi_s + 2\psi_0) + B_{13}\cos(3\psi_s + 3\psi_0) + B_{14}\sin(4\psi_s + 4\psi_0)\}]$

10) Then $E_2 = [1 - e'\cos(M_s + E_0 + E_1)]^{-1} M_2$

11) Place $E = M_s + E_0 + E_1 + E_2$ in the anomaly connections to find

$v = m_s + v_0 + v_1 + v_2$

12) Then $\psi_2 = \lambda_6 [A_2 v_2 + A_{21} v_1 \cos(M_s + v_0) + 2A_{22} v_1 \cos(2M_s + 2v_0)$

$+ A_{23}\sin(3M_s + 3v_0) + A_{24}\sin(4M_s + 4v_0)] - B_2^{-1}[-B_{21}\psi_1\sin(\psi_s + \psi_0)$

$+ 2B_{22}\psi_1\cos(2\psi_s + 2\psi_0) + B_{23}\cos(3\psi_s + 3\psi_0) + B_{24}\sin(4\psi_s + 4\psi_0)]$

Then $E = M_s + E_0 + E_1 + E_2$, $v = M_s + v_0 + v_1 + v_2$, $\psi = \psi_s + \psi_0 + \psi_1 + \psi_2$

13) $\rho = a(1 - \cos E) = (1 + e\cos v)^{-1} p$, $\eta = P + Q\sin\psi$

14) $\Omega' = \beta_3 - c^2 \alpha_3 \left(\frac{\mu}{a_0}\right)^{-\frac{1}{2}} (A_3 v + \sum_{k=1}^{4} A_{3k}\sin kv) + \alpha_3 \alpha_2^{-1} u^{\frac{1}{2}} (B_3\psi - \frac{3}{4}C_1 C_2 Q\cos\psi$

$+ \frac{3}{32} C_2{}^2 Q^2 \sin 2\psi)$

Then if sgn $\alpha_3 = \pm 1$ for direct or retrograde orbits respectively, the rectangular coordinates are

15) $X=(\rho^2+c^2)^{\frac{1}{2}}[H_1\cos\ \Omega'\cos\psi - H_1^{-1}(1-S)^{\frac{1}{2}}\mathrm{sgn}\alpha_3(H_2+H_3\sin\psi)\sin\ \Omega']$

16) $Y=(\rho^2+c^2)^{\frac{1}{2}}[H_1\sin\Omega'\cos\psi + H_1^{-1}(1-S)^{\frac{1}{2}}\sin\alpha_3(H_2+H_3\sin\psi)\cos\Omega']$

17) $Z=\rho\eta-\delta$

<u>Velocity Components</u>

18) $\dot{v} = \frac{a}{o}\ \frac{\mu(1-e^2)}{a_0}^{\frac{1}{2}}\ \frac{(\rho^2+A\rho+B)^{\frac{1}{2}}}{\rho^2+c^2\eta^2}$

19) $\dot{\rho} = \frac{e}{p}\ \rho^2\sin v\ \dot{v}$

20) $\dot{\psi} = \left(\frac{\mu p_0}{u}\right)^{\frac{1}{2}}(\rho^2+c^2\eta^2)^{-1}(1+C_1\eta-C_2\eta^2)^{\frac{1}{2}}$

21) $\dot{\Omega}' = -c^2\alpha_3\left(\frac{\mu}{a_0}\right)^{-\frac{1}{2}}(A_3+\sum_{k=1}^{4} kA_{3k}\cos kv)\dot{v}$

$$+\alpha_3\alpha_2^{-1}u^{\frac{1}{2}}(B_3+\frac{3}{4}\ C_1C_2Q\ \sin\psi+\frac{3}{16}\ C_2{}^2Q^2\cos\ 2\psi)\dot{\psi}$$

22) $\dot{X} = \frac{\rho\dot{\rho}}{\rho^2+c^2}\ X-Y\dot{\Omega}'+(\rho^2+c^2)^{\frac{1}{2}}[-H_1\sin\psi\cos\Omega'-H_1^{-1}(1-S)^{\frac{1}{2}}\mathrm{sgn}\alpha_3H_3\cos\psi\sin\Omega']\dot{\psi}$

23) $\dot{Y} = \frac{\rho\dot{\rho}}{\rho^2+c^2}\ Y+X\dot{\Omega}'+(\rho^2+c^2)^{\frac{1}{2}}[-H_1\sin\psi\sin\Omega'+H_1^{-1}(1-S)^{\frac{1}{2}}\mathrm{sgn}\alpha_3H_3\cos\psi\cos\Omega']\dot{\psi}$

24) $\dot{Z} = \eta\dot{\rho} + \rho Q\cos\psi\ \dot{\psi}$